4. Inverse Trigonometric functions

Two Marks:

1. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$.

$$y = \cos^{-1}(y_2)$$

$$\cos y = y_2 = \cos\frac{\pi}{3}$$

$$y = \frac{\pi}{3}$$

2. Find the value of $2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(y_2)$

$$2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(y_2)$$

$$= 2\left(\frac{\pi}{3}\right) + \left(\frac{\pi}{6}\right) = \left(\frac{4\pi + \pi}{6}\right) = \left(\frac{5\pi}{6}\right)$$

3. Find the domain of $f(x) = \sin^{-1}\left(\frac{|x| - 2}{3}\right) + \cos^{-1}\left(\frac{1 - |x|}{4}\right)$

Solution:

$$f(x) = \sin^{-1}\left(\frac{|x| - 2}{3}\right) + \cos^{-1}\left(\frac{1 - |x|}{4}\right)$$

(i)

$$-1 \le \frac{|x| - 2}{3} \le 1$$

$$-3 \le |x| - 2 \le 3$$

$$-1 \le |x| \le 5$$

$$0 \le |x| \le 5 ------- > (1)$$

$$-1 \le \frac{1 - |x|}{4} \le 1$$

$$-4 \le 1 - |x| \le 4$$

$$-5 \le -|x| \le 3$$

$$5 \ge |x| \ge -3$$

$$-3 \le |x| \le 5$$

$$0 \le |x| \le 5 --------- > (2)$$

From (1) and (2)

$$|x| \le 5$$

$$= -5 \le x \le 5$$

Domain $[-5, 5]$

4. Solve $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

$$\sin^{-1}\left(\frac{5}{x}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{12}{x}\right)$$
$$\sin^{-1}\left(\frac{5}{x}\right)=\cos^{-1}\left(\frac{12}{x}\right)$$
$$\sin^{-1}\left(\frac{5}{x}\right)=\cos^{-1}\left(\frac{12}{x}\right)=\theta$$
$$\left(\frac{5}{x}\right)=\sin\theta,\left(\frac{12}{x}\right)=\cos\theta$$

We know that,

$$\sin^2\theta+\cos^2\theta=1$$
$$\left(\frac{5}{x}\right)^2+\left(\frac{12}{x}\right)^2=1$$
$$\left(\frac{25}{x^2}\right)+\left(\frac{144}{x^2}\right)=1$$
$$\left(\frac{169}{x^2}\right)=1,\quad x^2=169,\quad x=\pm 13$$

5. Find the period and amplitude of $y=\sin\left(\frac{x}{3}\right)$

Solution:

$$y=\sin\left(\frac{x}{3}\right)$$
$$(y=\sin kx)$$
$$k=\frac{1}{3}$$

Period $\frac{2\pi}{|k|}=\frac{2\pi}{|1/3|}=\frac{2\pi}{1/3}=2\pi\times 3=6\pi$, Amplitude = 1.

6. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

We know that $\sin^{-1}(\sin\theta)=\theta$ if $\theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$$\frac{5\pi}{4}\notin\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

$$\sin^{-1}\left(\sin\frac{5\pi}{4}\right)=\sin^{-1}\left(\sin\left(\pi+\frac{\pi}{4}\right)\right)$$
$$=\sin^{-1}\left(\sin\frac{\pi}{4}\right)$$
$$=\frac{\pi}{4},\qquad \frac{\pi}{4}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

7. Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(-1)$

Solution:

$$\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(-1)\qquad \frac{1}{2}\in[-1,1]$$
$$=\frac{\pi}{3}-\frac{\pi}{2}=\frac{2\pi-3\pi}{6}=-\frac{\pi}{6}\qquad -1\in[-1,1]$$

8. Find the principal value of $\cos ec^{-1}\left(-\sqrt{2}\right)$.

Solution:

$$y=\cos ec^{-1}\left(-\sqrt{2}\right)$$
$$\cos ec\ y=\left(-\sqrt{2}\right)$$

Let $\sin y=\frac{-1}{\sqrt{2}}=\sin\left(\frac{-\pi}{4}\right),\qquad \frac{-\pi}{4}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$$y=\frac{-\pi}{4}$$
$$\cos ec^{-1}\left(-\sqrt{2}\right)=\frac{-\pi}{4}$$

9. Find the value, if it exists. If not, give the reason for non-existence $\sin^{-1}(\sin 5)$

Solution:

$$\pi\cong 3.142$$
$$3\pi\cong 3(3.142)$$
$$3\pi\cong 9.426$$
$$3\pi<10\qquad [5\ \textit{is very close to}\ \frac{3\pi}{2}$$
$$\frac{3\pi}{2}<5\qquad 5\notin\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$
$$\frac{3\pi}{2}-2\pi<5-2\pi$$
$$\frac{-\pi}{2}<5-2\pi$$
$$5-2\pi\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$
$$\sin^{-1}(\sin 5)=\sin^{-1}(\sin(5-2\pi))=5-2\pi$$

Alter:

$$5 \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \text{ But } \quad 5-2\pi \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin^{-1}(\sin 5) = \sin^{-1}(\sin(5-2\pi)) = 5-2\pi$$

10. Find the principal value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution:

$$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

(OR)

$$y = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$\tan y = \frac{-1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right)$$

$$y = \frac{-\pi}{6}$$

11. Find the principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$

Solution:

$$y = \cos^{-1}\left(\frac{-1}{2}\right)$$

$$\cos y = \frac{-1}{2} = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\cos y = \cos\frac{2\pi}{3}, \qquad \frac{2\pi}{3} \in [0, \pi]$$

$$y = \frac{2\pi}{3}$$

$$\therefore \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

(OR)

$$= \cos^{-1}\left(\frac{-1}{2}\right)$$

$$= \pi - \cos^{-1} 1/2$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{3\pi - \pi}{3}$$

$$= \frac{2\pi}{3}$$

12. Evaluate $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

Solution:

$$\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$$
$$=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

(OR)

$$\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$$
$$=\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=\frac{\sqrt{3}}{2}$$

3 Marks

1. For what value of *x*, the inequality $\frac{\pi}{2}<\cos^{-1}(3x-1)<\pi$ holds?

   <u>Solution</u>: $\frac{\pi}{2}<\cos^{-1}(3x-1)<\pi$

   Since $\cos\theta$ is a Decreasing function in $\left[\frac{\pi}{2},\pi\right]$

   $$\cos\frac{\pi}{2}>\cos\cos^{-1}(3x-1)>\cos\pi$$
   $$0>3x-1>-1$$
   $$\Rightarrow -1<3x-1<0$$
   $$\Rightarrow 0<3x<1$$
   $$\Rightarrow\ 0<x<1/3$$
   $$\Rightarrow\ x\in(0,1/3)$$

2. Find the value of $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)\right)+\sin^{-1}\left(\frac{4}{5}\right)$

   <u>Solution:</u> we know that $\cos^{-1}\theta+\sin^{-1}\theta=\frac{\pi}{2}$

   $$\cos\left(\cos^{-1}\left(\frac{4}{5}\right)\right)+\sin^{-1}\left(\frac{4}{5}\right)=\cos\frac{\pi}{2}=0$$

3. Find the domain of $\tan^{-1}\sqrt{9-x^2}$

   <u>Solution:</u>

   $$9-x^2\geq 0$$
   $$\Rightarrow 9\geq x^2$$
   $$\Rightarrow x^2\leq 9 \qquad \text{Domain } [-3,3]$$
   $$\Rightarrow x^2\leq 3^2$$
   $$\Rightarrow -3\leq x\leq 3$$

4. Find the value of $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{-1}{2}\right)\right)$

   <u>Solution:</u>

$$\tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{-1}{2}\right)\right)$$

$$=\tan\left(\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$==\tan\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\tan\frac{\pi}{2}=\infty$$

5. Find the value of $\sin^{-1}\left(\cos\left((\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)$

Solution:

$$\sin^{-1}\left(\cos\left((\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)$$

$$=\sin^{-1}\left(\cos\frac{\pi}{3}\right)$$

$$=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

6. Prove that $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}},\quad -1<x<1$

Proof:

$$y=\tan\left(\sin^{-1}x\right)=\tan\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)=\frac{x}{\sqrt{1-x^2}}\quad -1<x<1$$

7. Prove that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$

Proof:

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$$

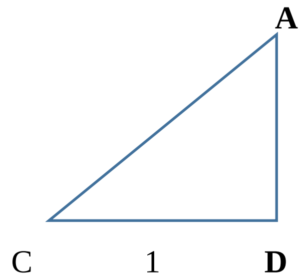
$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{(3+2)}{6}}{1-\frac{1}{6}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

8. Solve $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left(\cot^{-1}\left(\frac{3}{4}\right)\right)$

Solution:



$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Given that

$$\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$$

$$\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\sin^{-1}\left(\frac{4}{5}\right)\right\}$$

$$\frac{1}{\sqrt{1+x^2}}=\left(\frac{4}{5}\right)$$

Squaring on both sides, $\frac{1}{1+x^2}=\left(\frac{16}{25}\right)$

$$1+x^2=\left(\frac{25}{16}\right)$$

$$x^2=\left(\frac{25}{16}\right)-1$$

$$x^2=\left(\frac{9}{16}\right)$$

$$x=\pm\left(\frac{3}{4}\right)$$

9. Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

Solution:

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{(x-1)(x+2)+(x+1)(x-2)/x^2-4}{1-\frac{x^2-1}{x^2-4}}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{(x-1)(x+2)+(x+1)(x-2)}{(x^2-4)-(x^2-1)}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right)=\frac{\pi}{4}$$

$$\frac{2x^2-4}{-3}=\tan\frac{\pi}{4}$$

$$\frac{2x^2-4}{-3}=1$$

$$2x^2-4=-3$$

$$2x^2=-3+4$$

$$2x^2=1$$

$$x^2=1/2$$

$$x=\pm\frac{1}{\sqrt{2}}$$

5 Marks

1. Find the value of $\tan^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)$

   Solution:

   $$\tan^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)$$

   $$=-\tan^{-1}(1)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$$

   $$=\frac{\pi}{4}+\frac{\pi}{3}-\frac{\pi}{6}=\frac{-\pi}{4}+\frac{\pi}{6}=\frac{-3\pi+2\pi}{6}=\frac{\pi}{12}$$

2. Prove that $\frac{\pi}{2}\le\sin^{-1}x+2\cos^{-1}x\le\frac{3\pi}{2}$

Solution: $\sin^{-1}x + 2\cos^{-1}x = \sin^{-1}x + \cos^{-1}x + \cos^{-1}x = \frac{\pi}{2} + \cos^{-1}x$

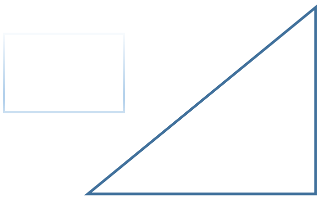
We know that $0 \le \cos^{-1}x \le \pi$

$$0 + \frac{\pi}{2} \le \frac{\pi}{2} + \cos^{-1}x \le \frac{\pi}{2} + \pi$$

$$\frac{\pi}{2} \le \sin^{-1}x + 2\cos^{-1}x \le \frac{3\pi}{2}$$

3. Evaluate $\sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \sec^{-1}\left(\frac{5}{4}\right)\right]$

Solution:



4

$$\sin^{-1}\left(\frac{3}{5}\right) = \sec^{-1}\left(\frac{5}{4}\right) = \theta$$

$$= \sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \sec^{-1}\left(\frac{5}{4}\right)\right] = \sin(\theta + \theta) = \sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

4. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ and $0 < x, y, z < 1$. Show that $x^2 + y^2 + z^2 + 2xyz = 1$

Solution:

$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right] = \pi - \cos^{-1}z$

$\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right] = \cos\left(\pi - \cos^{-1}z\right)$

$\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right] = -\cos\cos^{-1}(z)$

$$xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

Squaring on both sides,

$(xy+z)^2 = (1-x^2)(1-y^2)$

$x^2 + y^2 + z^2 + 2xyz = 1$

Alter:

$$\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi=\frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{3}$$

$$\cos^{-1}x=\frac{\pi}{3},\ \cos^{-1}y=\frac{\pi}{3},\ \cos^{-1}z=\frac{\pi}{3}$$

$$x=\frac{1}{2},\ y=\frac{1}{2},\ z=\frac{1}{2}$$

$$x^2+y^2+z^2+2xyz=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\left(\frac{3}{4}+\frac{1}{4}\right)=1$$

5. If $a_1,a_2,a_3,\ldots\ldots\ldots\ldots\ldots a_n$ is an A.P. with common difference 'd' prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots\ldots\ldots\ldots+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_2}$$

Solution:

Since $a_1,a_2,a_3,\ldots\ldots\ldots\ldots\ldots a_n$ is an A.P.

$$a_2-a_1=a_3-a_2=\ldots\ldots\ldots\ldots\ldots=.a_n-.a_{n-1}=d$$

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)=\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)=\tan^{-1}a_2-\tan^{-1}a_1$$

..

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..

Now .

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$$\tan^{-1}\left(\frac{d}{1+a_{n1}a_{n-1}}\right)=\tan^{-1}a_n-\tan^{-1}a_{n-1}$$

..

$$LHS=\tan\left[\tan^{-1}a_2-\tan^{-1}a_1+\tan^{-1}a_3-\tan^{-1}a_2+\ldots\ldots\tan^{-1}a_n-\tan^{-1}a_{n-1}\right]$$

$$=\tan\left[\tan^{-1}a_n-\tan^{-1}a_{n-1}\right]==\tan\left[\tan^{-1}\frac{a_n-a_1}{1+a_1a_n}\right]=\frac{a_n-a_1}{1+a_1a_n}$$

6. Fin the value of $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

Solution:

$$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2},\quad 2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$$

Given that $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

$$= \tan\left[\frac{1}{2}\left(2\tan^{-1}a\right)+\frac{1}{2}\left(2\tan^{-1}a\right)\right]$$
$$= \tan\tan\left[\left(\tan^{-1}a\right)+\left(\tan^{-1}a\right)\right]$$
$$= \tan\tan\left[\left(2\tan^{-1}a\right)\right]$$
$$= \tan\left[\left(\tan^{-1}\frac{2a}{1-a^2}\right)\right]$$
$$= \frac{2a}{1-a^2}$$

7. Solve $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$

Solution: $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$
$$2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$
$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{6}$$
$$\left(\frac{2x}{1-x^2}\right)=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$
$$2x(\sqrt{3})=1-x^2$$
$$x^2+x(2\sqrt{3})-1=0$$
$$x^2+x(2\sqrt{3})=1$$
$$(x+\sqrt{3})^2-(\sqrt{3})^2=1$$

$$(x+\sqrt{3})^2-3=1$$
$$(x+\sqrt{3})^2=4$$
$$(x+\sqrt{3})=\pm 2$$
$$x=\pm 2-\sqrt{3}$$

8. Find the domain of $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Solution: For any real number x

$$\left(\frac{x^2+1}{2x}\right) \neq 0$$

$$\left(\frac{x^2+1}{2x}\right) \in [-1, 1] - \{0\}$$

$$-1 \leq \frac{x^2+1}{2x} < 0, \quad x < 0$$

$mul\ 2x$

$$-2x \geq x^2+1$$

$$\Rightarrow 0 \geq x^2+1+2x$$

$$\Rightarrow (x+1)^2 \leq 0$$

This is possible only if *x+1=0*, $x = -1$

$$0 \leq \frac{x^2+1}{2x} < 1, \quad x > 0$$

$mul\ 2x$

$$0 < x^2+1 \leq 2x$$

$$\Rightarrow x^2+1+2x \leq 0$$

$$\Rightarrow (x-1)^2 \leq 0$$

This is possible only if *x-1=0*, $x = 1$.

Domain ={-1,1}

<u>Alter:</u>

$$y = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$$

$$\sin y = \frac{x^2+1}{2x}$$

$$2x \sin y = x^2+1$$

$$x^2 - 2x \sin y + 1 = 0$$

$$x^2 - (2\sin y)x + \sin^2 y + \cos^2 y = 0$$

$$(x - \sin y)^2 = -\cos^2 y$$

$$(x - \sin y) = \pm\sqrt{-\cos^2 y} = \pm i \cos y$$

$$x = \sin y = \pm i \cos y$$

Since $x \in R, \quad \cos y = 0 \Rightarrow y = \pm\frac{\pi}{2}$

When $y = \frac{\pi}{2}$ $\qquad$ $y = \frac{-\pi}{2}$

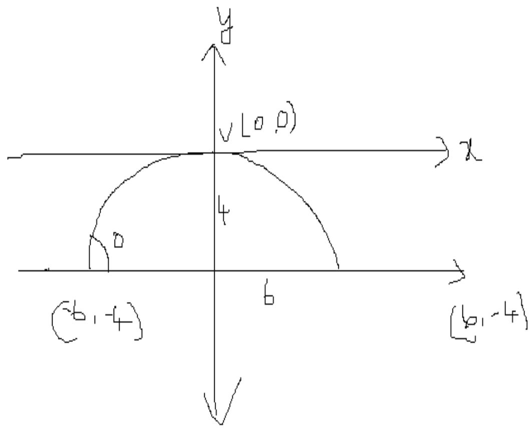
$x = 1$ $\qquad$ $x = -1$

Domain ={1,-1}

UNIT-5 (5Marks)

1. On lighting a rocker cracker it gets projected in a parabolic path and reaches the ground 12mts away from the starting point. Find the angle of projection.

Solution:



The equation is $x^2 = -4ay$ --------------→(1)

The point (6,-4) lies on the parabola $6^2 = -4a(-4)$

$$36 = 16a \Rightarrow a = \frac{36}{16} = \frac{9}{4}$$

$$(1) \Rightarrow x^2 = -4\left(\frac{9}{4}\right)y$$

$$x^2 = -9y$$

Differentiate with respect to 'x',

$$2x = -9\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{-9} = \frac{-2x}{9}$$

At (-6,-4)

$$\frac{dy}{dx} \Rightarrow \frac{-2(-6)}{9} = \frac{12}{9} = \frac{4}{3}$$

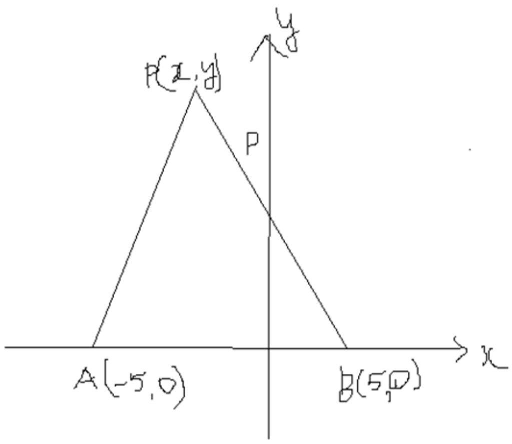
$$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$.

2. Points A and B are 10km apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6km closer

to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Solution:



Given PB-PA=6

$$\sqrt{(x-5)^2+(y-0)^2}-\sqrt{(x+5)^2+(y-0)^2}=6$$

$$\sqrt{(x-5)^2+y^2}-\sqrt{(x+5)^2+y^2}=6$$

Squaring on both sides $(x-5)^2+y^2=(x+5)^2+y^2+36+12\sqrt{(x+5)^2+y^2}$

$$x^2+25-10x+y^2=x^2+25+10x+y^2+36+12\sqrt{(x+5)^2+y^2}$$

$$-20x-36=12\sqrt{(x+5)^2+y^2}$$

Dividing by -4, $\quad 5x+9=-3\sqrt{(x+5)^2+y^2}$

Squaring,

$$25x^2+81+90x=9\left[(x+5)^2+y^2\right]=9[x^2+10x+25+y^2]=9x^2+90x+225+9y^2$$

$$16x^2-9y^2=144$$

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

Which is a hyperbola.

3. A road bridge over an irrigation canal have two semi-circular vents each with a span of *20m* and the supporting pillars of width *2m*. Use Fig 5.16 to write the equations that model the arches.
Solution:

Let O1, O2 be the centres of the two semi-circular vents.
First vent with centre O1, (12,0) and radius r = 10.
The equation of first semicircle $(x-h)^2 + (y-0)^2 = 10^2$

$$x^2 - 24x + 44 + y^2 = 0, \quad y > 0$$

Second vent with centre O2 (34,0) and radius r = 10.
The equation of second semi-circle $(x-34)^2 + (y-0)^2 = 10^2$

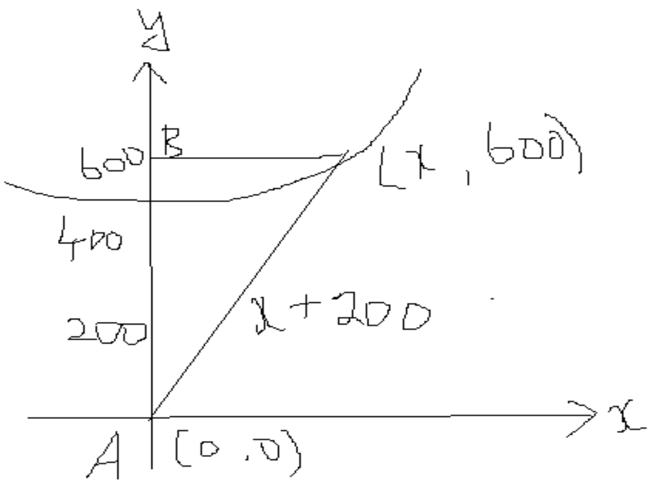
$$x^2 - 68x + 1056 + y^2 = 0, \quad y > 0$$

4. Two coast guard stations are located 600km apart at points A(0,0) and B(0,600). A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Solution:

Since the centre is located at (0,300), midway between the thwo foci, which are the coast guard stations, the equation is

$$\frac{(y-300)^2}{a^2} - \frac{(x-0)^2}{b^2} = 1 \text{---------------------(1)}$$



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a=Distance between centre and vertices *=400-300*

a=100

$c^2 = a^2 + b^2$

$b^2 = c^2 - a^2 = (300)^2 - (100)^2 = 90000 - 10000 = 80000$

[$\therefore c$ is distance between focus and centre c=600-300=300]

$$\frac{(y-300)^2}{10000} - \frac{(x-0)^2}{80000} = 1$$

5. Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is *14m* above the vertex of the parabola. The hyperbola's second focus F_2 is *2m* above the parabola's vertex. The vertex of the hyperbolic mirror is 1m below F_1. Position a

coordinate system with the origin at the centre of the hyperbola and with the foci on the *y*-axis. The find the equation of the hyperbola.

Solution:

Let V1 be the vertex of the parabola and V2 be the vertex of the hyperbola.

$$\overline{F_1F_2} = 14 - 2 = 12m$$
$$2C = 12$$
$$c = 6$$

The distance of centre to the vertex of the hyperbola is $a = 6 - 1 = 5$,

$$b^2 = c^2 - a^2 = 36 - 25 = 11$$

The equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{11} = 1$.

6. Find the vertex focus, directrix and length of the latus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Solution:

$$x^2 - 4x - 5y - 1 = 0$$
$$x^2 - 4x - 5y = 1$$
$$x^2 - 4x + 4 = 5y + 1 + 4$$
$$(x-2)^2 = 5y + 5 = 5(y+1) ---------- (1)$$

Comparing with $(x-h)^2 = 4a\,(y-k)$

Here $4a = 5,$ $a = 5/4,\ \ h = 2,\ \ k = -1$ Vertex = *A(h,k)=(2,-1)*

Focus =*s (0+h, a+k)=s(0+2,5/4-1)=s(2, ¼)*

Equation of directrix $y = k - a = -1 - 5/4 = -9/4$
$4y + 9 = 0$

Length of latus rectum $4a = 5$ units.

7. Find the foci, verticles and length of major and minor axis of the conic $4x^2 + 36y^2 + 40x - 288y + 532 = 0$

$$4x^2 + 36y^2 + 40x - 288y = -532$$
$$4(x^2 + 10) + 36(y^2 - 8y) = -532$$
$$4(x^2 + 10x + 25 - 25) + 36(y^2 - 8y + 16 - 16) = -532$$
$$4\left[(x+5)^2 - 25\right] + 36\left[(y-4)^2 - 16\right] = -532 + 100 + 576$$
$$4(x+5)^2 + 36(y-4)^2 = 144$$
$$\div 144,\quad \frac{(x+5)^2}{36} + \frac{(y-4)^2}{4} - = 1$$

The major axis parallel to x-axis.

Here $h=-5, k=4,\ a^2=36,\ b^2=4,$

$$a=6,\ \ b=2$$

The centre = c (h,k)=c (-5,4)

Vertices: $A^{'}(h-a,k)=A^{'}(-5-6,4)=A^{'}(-11,4)$

$A(h+a,k)=A(-5+6,4)=A^{'}(1,4)$

Now $c^2=a^2-b^2=36-4=32$

$c=\pm 4\sqrt{2}$

Foci $S^{'}(h-c,k)=S^{'}(-5-4\sqrt{2},4)$

$S(h+c,k)=S(-5+4\sqrt{2},4)$

Length of major axis $=2a=2(6)=12$ units

Length of minor axis $=2b=2(2)=4$ units

8. For the ellipse $4x^2+y^2+24x-2y+21=0$ .Find the centre, vertices and the foci. Also prove that the length of latus rectum is 2.

Solution:

$$4x^2+y^2+24x-2y+21=0$$

$$4x^2+24x+y^2-2y+21=0$$

$$4(x^2+6x)+(y^2-2y)=-21$$

$$4(x^2+6x+9-9)+(y^2-2y+1-1)=-21$$

$$4\left[(x+3)^2-9\right]+\left[(y-1)^2-1\right]=-21$$

$$4(x+3)^2+(y-1)^2=-21+36+1$$

$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}-=1$$

Major axis parallel to y-axis.

Here $h=-3, k=1,\ a^2=16,\ b^2=4,$

$$a=4,\ \ b=2$$

$$c^2=a^2-b^2=16-4=12$$

$$c=\pm 2\sqrt{3}$$

Vertices: $A^{'}(h,k-a)=A^{'}(-3,1-4)=A^{'}(-3,3)$

$A(h,k+a)=A(-3,1+4)=A^{'}(-3,5)$

Foci $S^{'}(h,k-c)=S^{'}(-3,1-2\sqrt{3})$

$S(h,k+c)=S(-3,1+2\sqrt{3})$

Length of latus rectum $\frac{2b^2}{a}=\frac{2(4)}{4}=2$ units.

9. Find the centre, foci and eccentricity of the hyperbola $11x^2-25y^2-44x+50y-256=0$

Solution:

$11x^2-25y^2-44x+50y=256$

$11(x^2-4x)-25(y^2-2y)=256$

$11(x^2-4x+4-4)-25(y^2-2y+1-1)=256$

$11(x-2)^2-25(y-1)^2=256+44-25$

$11(x-2)^2-25(y-1)^2=275$

$$\frac{(x-2)^2}{25}-\frac{(y-1)^2}{11}-=1$$

Transverse axis parallel to x-axis

Here $h=2, k=1, a^2=25, b^2=11,$
$a=5$

Now $c^2=a^2+b^2=25+11=36$
$c=\pm 6$

The centre = c (h,k)=c (2,1)

Foci $S^{'}(h-c,k)=S^{'}(2-6,1)=S'(-4,1)$
$S(h+c,k)=S(2+6,1)=S(8,1)$

Eccentricity = *e=c/a=6/5*

10. Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is ar the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:

The equation $x^2=-4ay$---------------------(1)

The point [3,-2.5] lies on the parabola

$$3^2=-4a(-2.5)$$

$$a=\frac{9}{10}$$

$$x^2=-4\left(\frac{9}{10}\right)y \text{------------------(2)}$$

The point [x,-7.5] lies on the parabola $x_1{}^2=-4\times\left(\frac{9}{10}\right)\times -7.5=30\times\left(\frac{9}{10}\right)=9\times 3$

$$x_1=3\sqrt{3}m$$

The water strikes the ground $3\sqrt{3}m$ beyond the vertical line.

11. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the

edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?

Solution:

Given $AA'' = 2a = 16$ given b=5

$a = 8$

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{64} + \frac{y^2}{25} = 1 \text{---------------------(1)}$$

P(x,4) lies on ellipse

$$\frac{{x_1}^2}{64} + \frac{4^2}{25} = 1$$

$$\frac{{x_1}^2}{64} = 1 - \frac{16}{25} = \frac{9}{25}$$

$$ {x_1}^2 = 64 \times \frac{9}{25}$$

$$x_1 = \frac{8 \times 3}{5} = 4.8$$

Required wide $= = 2x_1 = 2 \times 4.8 = 9.6m$

12. At a water fountain, water attains a maximum height of *4m* at horizontal distance of *0.5m* from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of *0.75m* from the point of origin.

Solution:

The equation of parabola is $(x-h)^2 = -4a(y-k)$ ------------------------(1)

Here the vertex is (0.5,4)

(1) => $(x-0.5)^2 = -4a(y-4)$ --(2)

O (0,0) is a point on the parabola

$(0-0.5)^2 = -4a(0-4)$

(2) => $0.25 = 4a(4)$

$$4a = \frac{0.25}{4}$$

(2) => becomes $(x-0.5)^2 = -\frac{0.25}{4}(y-4)$ -------------------------(3)

$P(0.75, y_1)$ lies on parabola

$$(0.75-0.5)^2 = -\frac{0.25}{4}(y_1 - 4)$$

$$(0.25)^2 = -\frac{0.25}{4}(y_1 - 4)$$

$$y_1 - 4 = -\frac{4(0.25)^2}{0.25} == 4(0.25) = -1$$

$$y_1 = -1 + 4 = 3$$

Height of the water at a horizontal distance of *0.75m* is *3m*.

13. An engineer designs a satellite dish with a parabolic cross section. The dish is 5m wide at the opening and the focus is placed 1.2m from the vertex.
    (a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.
    (b) Find the depth of the satellite dish at the vertex.

Solution:

(a) Vertex is V(0,0)

Equation of parabola is $y^2 = 4ax$-------------------(1)

VF = a = *1.2m*

(1) => $y^2 = 4(1.2)x$

$y^2 = 4.8x$ −−−−−−−−−−−−−−(2)

(b) $P(x_1, 2.5)$ lies on parabola

(2) => $(2.5)^2 = 4.8x_1$

$x_1 = \frac{(2.5)^2}{4.8} = 1.3m$

Depth of the satellite dish at the vertex is *1.3m*

14. Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical cables are to be spaced every 6m along this portion of the roadbed. Calculate the length of first two of these vertical cables from the vertex.

Solution:

Vertex is V(0,3)

Equation of parabola is $(x-h)^2 = 4a(y-k)$

$(x-0)^2 = 4a(y-3)$

$x^2 = 4a(y-3)$ −−−−−−−(1)

P(30,16) lies on parabola $= 4a(16-3)$

$4a = \frac{30 \times 30}{13}$

$x^2 = \frac{30 \times 30}{13}(y-3)$

$Q(6, y_1)$ lies on parabola

(2) => $6^2 = \frac{30 \times 30}{13}(y_1 - 3)$

$36 = \frac{30 \times 30}{13}(y_1 - 3)$

$y_1 - 3 = \frac{36 \times 13}{30 \times 30}$

$y_1 = 3.52m$

$R12, y_2)$ lies on parabola

$$(12)^2 = \frac{30\times 30}{13}(y_2 - 3)$$

(3) => $y_2 - 3 = \frac{144\times 13}{30\times 30}$

$$y_2 = \frac{144\times 13}{30\times 30} + 3$$

$$y_2 = 5.08m$$

Hence the length of first two vertical cables are 3.52m and *5.08m*.

15. Cross section of a nuclear cooling tower is I n shape of a hyperbola with equation $\frac{x^2}{30^2} + \frac{y^2}{44^2} = 1$. The tower is 150m tall and the distance from the top of the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top the base of the tower.

Solution:

Given *A+2A=150*

3A=150

A=50

Distance from top to centre is *50m*. Distance from base to centre is *100m*.

Given equation of the hyperbola is $\frac{x^2}{30^2} + \frac{y^2}{44^2} = 1$------------------------(1)

$P(x_1, 50)$ lies on parabola

$$\frac{x_1{}^2}{30^2} - \frac{50^2}{44^2} = 1$$

$$\frac{x_1{}^2}{30^2} = 1 + \frac{50^2}{44^2}$$

(1) => $\frac{x_1{}^2}{30^2} = 2.291$

$$x_1{}^2 = 30^2 \times 2.291$$

$$x_1 = 45.41m$$

$P(x_2, 100)$ lies on parabola (1) = >

$$\frac{x_{21}{}^2}{30^2} - \frac{5010}{44^2} = 1$$

$$\frac{x_2{}^2}{30^2} = 1 + \frac{100}{44^2}$$

$$\frac{x_2{}^2}{30^2} = 6.16$$

$$x_2{}^2 = 30^2 \times 6.16$$

$$x_2 = 74.45m$$

Diameter of the top $= 2x_1 = 2(45.41) = 90.82m$

Diameter of the base $= 2x_2 = 2(74.45) = 148.9m$

16. A rod of length 1.2m moved with its ends always touching the co-ordinate axes. The locus of a point P on the rod. Which is 0.3m from the end in contact with x-axis is an ellipse. Find the eccentricity.

Solution:

Let AB be the rod and $P(x_1, y_1)$ be a point on the rod such that AP=0.3m

Draw $PD \perp x - axis$ and $PC \perp y - axis$

Let $\angle PAD = \angle BPC = \theta$

From right angled $\Delta BPC,$ $\quad \cos\theta = \frac{x_1}{0.9}$

From right angled $\Delta PAD,$ $\quad \sin\theta = \frac{y_1}{0.3}$

$$\cos^2\theta + \sin^2\theta = 1$$

We know that $\left(\frac{x_1}{0.9}\right)^2 + \left(\frac{y_1}{0.3}\right)^2 = 1$

$$\frac{x_1{}^2}{0.81} + \frac{y_1{}^2}{0.09} = 1$$

The locus of $P(x_1, y_1)$ is $\frac{x_1{}^2}{0.81} + \frac{y_1{}^2}{0.09} = 1$

$$a^2 = 0.81$$

$$b^2 = 0.09$$

Here $= \sqrt{1 - \frac{b^2}{a^2}}$

$$= \sqrt{1 - \frac{0.09}{0.81}}$$

$$= \sqrt{1 - \frac{9}{81}} == \sqrt{\frac{72}{81}} == \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$